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Algorithms for solution of fixed point and equilibrium problems in a Banach space

Enyinnaya Ekuma-Okereke[1*] and Abel Okojunu[1]
[1]Department of Mathematics, Federal University of Petroleum Resources, Effurun, Nigeria.
[*]Corresponding author: ekuma.okereke@fupre.edu.ng
abelokojunu@gmail.com

Abstract
This paper considers two new algorithms of inertial hybrid type for finding common fixed point problems involving finite family of Bregman strongly nonexpansive mappings which also finds the common solution of finite system of equilibrium problems. We prove strong convergence results for the constructed algorithms. We apply our algorithms in finding common solutions of convex feasibility problems. We also demonstrate numerical experiments to verify the theoretical assertions and properties. We observe the performance results of our algorithms with the inertial component known to improve and speed up convergence. From the generated data, we observed that the solutions for our methods are significantly similar for any well-chosen standard tolerance rate. The results obtained further demonstrate the effectiveness, applicability and convergence of our algorithms in Banach spaces. Our results significantly improves, extends many cited works in the literature.

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1. Introduction

In a Hilbert space H, and for any given domain $dom(G)$ and range $ran(G)$ of a mapping $G: dom(G) \to ran(G)$, we say that $G: dom(G) \to ran(G)$ is nonexpansive with respect to the norm function $\|.\|: dom(G) \to R$ if

$$\|Gu - Gz\| \leq \|u - z\|, \ \forall u, z \in dom(G), \quad (1)$$

where, R is the set of real numbers. Also the map $G: dom(G) \to ran(G)$ is quasi-nonexpansive with respect to the norm function $\|.\|: dom(G) \to R$ if

$$\|Gu - p\| \leq \|u - p\|, \quad \forall u \in dom(G), p \in Fix(G), \quad (2)$$

where $Fix(G) = \{p \in dom(G): Gu = u\}$ is the set of fixed point of map $G: dom(G) \to ran(G)$. So every nonexpansive mapping is quasi-nonexpansive with a fixed point. Examples of quasi-nonexpansive mappings which are not nonexpansive mappings can be found in Ofoedu et al (2009). An element $x^* \in dom(G)$ is called asymptotic fixed point of the map $G: dom(G) \to ran(G)$ if sequence $\{u_n\}$ is contained in $dom(G)$ and converges weakly to x^* so that $\|u_n - Gu_n\| \to 0$ as $n \to \infty$, see Chang (2010). Throughout this paper, we denote the asymptotic fixed point of a mapping $G: dom(G) \to ran(G)$ by $\hat{F}ix(G)$. Recall as pointed out by Chidume (2009) that nonexpansive criterion is a useful criterion in analysis in that it is intimately connected with the monotonicity criterion and it also appears in application as the transition operator for initial value problems of differential inclusions.

In this paper, R is the set of real numbers, N is the set of natural numbers We denote a Banach space by X. The dual space of X is denoted by X^*. We denote by K a nonempty, closed and convex subset of a Banach space X. We let $\|.\|: X \to R$ represent the norm function and $\langle .,. \rangle$ represent the duality pair of X and X^*. Let $d_h: dom h \times \text{int}(dom h) \to [0, \infty)$ represent a distance function induced by a convex function $h : X \to (-\infty,+\infty]$. Let $dom h = \{u \in X : h(u) < +\infty\}$ and $\text{int}(dom h)$ represent the domain and interior domain of a convex function $h : X \to (-\infty,+\infty]$ respectively. The conjugate function generally known as Fenchel conjugate function, (see Reich and Sabach (2010)) of the convex function $h : X \to (-\infty,+\infty]$ is the function $h^* : X^* \to (-\infty,+\infty]$ defined by:

$$h^*(x^*) = \sup\{\langle u, x^*\rangle - h(u) : u \in X\}. \quad (3)$$

The convex function $h : X \to (-\infty,+\infty]$ is called G$\hat{a}$teaux differentiable at u if $\lim_{s\to 0^+} \frac{h(u+sz) - h(u)}{s} = h^\circ(u,z)$ exists for any z in X. This shows that $h^\circ(u,z)$ coincides with the gradient of $h : X \to (-\infty,+\infty]$ denoted by $\nabla h(u)$. Thus, $h^\circ(u,z) = \nabla h(u)$ for all $u \in \text{int}(dom h)$, see Reich and Sabach (2010). Supposing that the convex function $h : X \to (-\infty,+\infty]$ is G$\hat{a}$teaux differentiable at $u \in \text{int}(dom h)$, then the function $d_h : dom h \times \text{int}(dom h) \to [0,\infty)$ defined by:

$$d_h(z,u) = h(z) - h(u) - \langle \nabla h(u), z - u\rangle \quad (4)$$

is called Bregman distance function induce by the convex function $h : X \to (-\infty,+\infty]$ see Bregman (1967). It has now been studied very well by many authors (see Chen et al. (2011), Naraghirad and Yao (2013), Ekuma-Okereke and Oladipo (2020) and volume of other authors). It is important to remark that the function $d_h : dom h \times \text{int}(dom h) \to [0,\infty)$ defined by equation (4) does not satisfy the classical properties of a metric function (usual distance), that is, the symmetric and triangle inequality properties of a metric function are not satisfied. So $d_h : dom h \times \text{int}(dom h) \to [0,\infty)$ is not a metric function in the usual sense of it, but it is easy to see that it satisfies the following nice properties (P1)-(P6):

(P1) The function $d_h(.,u)$ is convex with respect to the first variable,

(P2) $d_h(u,u) = 0$ if and only if $z = u$,

(P3) $d_h(z,u) > 0$ if and only h is strictly convex and $z \neq u$,

(P4) $d_h(z,u) = d_h(z,v) + d_h(v,u) + \langle z - v, \nabla h(v) - \nabla h(u)\rangle$,

(P5) $d_h(u,v) + d_h(v,u) = \langle u - v, \nabla h(u) - \nabla h(v)\rangle$,

(P6) $d_h(u,v) \le \| u \|\| \nabla h(u) - \nabla h(v) \| + \| v \|\| \nabla h(u) - \nabla h(v) \|$.

Now given the convex function $h : X \to (-\infty,+\infty]$ as a square norm with $u \in X$, such that X is smooth, then we see that $\nabla h(u) = 2Ju$, where $J : X \to X^*$ is called the normalized duality mapping defined by set $J(x) = \{x^* \in X^* : \langle x, x^*\rangle = \| x \| = \| x^* \|\}$, so that equation (4) reduce to

$$\phi(z,u) = \| z \|^2 - 2\langle Ju, z\rangle + \| u \|^2 \quad (5)$$

which is known as the Lyapunov functional and has been successfully used by so many authors like Chidume et al. (2018), Chang et al. (2010) and many others.

In this paper, we shall also need the following functions.

The function $h : X \to (-\infty,+\infty]$ is Legendre according to Bauschke and Borwein (1997), Bauschke (2001), Butnariu et al. (2006), if the following hold

(1) $\text{int}(dom h)$ is non-void, h is differentiable on $\text{int}(dom h)$ with $dom h = \text{int}(dom h)$,

(2) $\text{int}(dom h^*)$ is non-void, h^* is differentiable on $\text{int}(dom h^*)$ with $dom h^* = \text{int}(dom h^*)$.

We remark that with $h : X \to (-\infty,+\infty]$ a Legendre function, and X reflexive, then ∇h is a bijection which satisfies

$\nabla h = \left(\nabla h^{*}\right)^{-1}$,

$range\nabla h = domain\nabla h^{*} = \text{int}(dom\, h^{*})$.

If $h : X \to (-\infty, +\infty]$ have a single value and X is smooth and strictly convex, then $J = \nabla h$. Given $h(u) = t^{-1} \| u \|^{2}$, $t \in (1, \infty)$, then we have an important example of a Legendre function which was given by Bauschke and Borwein (1997) and have been used by so many authors.

The modulus of total convexity of h at $u \in \text{int}(dom\, h)$ is the function $W_h(u,.) : \text{int}(dom\, h) \times [0,\infty) \to [0,\infty)$ defined by

$$W_f(u,s) = \inf\{d_h(z,u) : z \in dom\, h, \| z - u \| = s\}. \quad (6)$$

If $W_f(.,s)$ is positive, then $h : X \to (-\infty, +\infty]$ becomes totally convex at u for positive value of s. Totally convex function is important in the convergence of a sequence in the Bregman distance, see Butnariu and Iusem (2010) for more information.

The Bregman distance function denoted throughout this paper by $d_h(.,.)$ induce by a convex function $h : X \to (-\infty, +\infty]$ instead of the classical norm function $\|.\|$ is effective to analyze the convex feasibility algorithms in optimization problem. Over the years, this process has turned out to be a useful and implementable tool for many researchers and institutions in designing and analyzing various types of algorithms towards finding the feasible point in problems such as fixed point theory, convex feasibility problem, equilibrium problems, Nash equilibrium problems and image restoration problems, just to mention but a few (see Zegeye (2014) and the references contained in it). Among other reasons for the use of this tool is the fact that outside Hilbert space, some important mappings and operators in application are not nonexpansive, for example metric projection mapping and resolvent operator of a monotone operator (see Reich and (2011) for more information). To this end, construction of algorithms for solving a common feasibility point in a Banach space is sort for. This common feasibility problem is for a fixed point and equilibrium problems involving some class of nonlinear nonexpansive mappings corresponding to the Bregman distance function and equilibrium bifunctions.

Since the Bregman distance function $d_h(.,.)$ is not symmetric, it suffices to state that we have directional (left and right) definitions of this function. Furthermore, from the definition of the function $d_h(.,.)$, we shall call $d_h(.,u)$ and $d_h(u,.)$ the left and right Bregman distances with respect to first and second variables respectively. It is easy to see that $d_h(u,.)$ is not convex with respect to the second variable. On the other hand, $d_h(.,u)$ is convex with respect to the first variable, hence the study of left Bregman nonexpansive mapping now defined below.

A map $G : K \to \text{int}(dom\, h)$, is Left Bregman quasi nonexpansive (L-BQNE) if $d_h(p, Gu) \leq d_h(p,u)$, $\forall u \in K, \forall p \in Fix(T)$. A map $T : K \to \text{int}(dom\, f)$, is Left Bregman strongly-quasi nonexpansive mapping (L-BSQNE) if it is Left Bregman quasi nonexpansive and if whenever $\{x_n\}_{n=1}^{\infty} \subset K$ is bounded, and

$$\lim_{n\to\infty} d_h(p, u_n) - d_h(p, Gu_n) = 0$$

it follows that

$$\lim_{n\to\infty} d_h(Gu_n, u_n) = 0.$$

A map $T : K \to \text{int}(dom\, f)$, is Left Bregman relatively nonexpansive (L-BRNE) if

$$d_h(p,Gu) \le d_h(p,u),\ \forall u \in K, \forall p \in Fix(T) = \hat{F}ix(T).$$

For more information on the definitions above (see Martin-Marquez et al. (2013), Zegeye (2014) and the references contained in them). Notably, interesting works has been done on some left classes of Bregman nonexpansive-type mappings. This achievement is connected to the fact that $d_h(.,Gu)$ is convex with respect to the first variable. We give an example of a Bregman strongly nonexpansive mapping.

Example 1.1 Let $X = l^2 = H$, where

$$l^2 = \left\{ \bar{x} = (x_1, x_2, \cdots) : \sum_{n=1}^{\infty} \| x \|^2 < \infty \right\},$$

$$\| \bar{x} \| = \left(\sum_{n=1}^{\infty} |x_n|^2 \right)^{\frac{1}{2}},\ \forall \bar{x} \in l^2,$$

$$\langle \bar{x}, \bar{y} \rangle = \sum_{n=1}^{\infty} x_n y_n.$$

Let $\{x_n\}_{n=1}^{\infty} \subset X$ be a sequence defined by $x_n = (x_1, x_2, \cdots) = \left(1, \frac{1}{2}, \cdots\right)$. It is clear that $x_n \in l^2$. It is also clear that x_n converges weakly to 0. Indeed for any $\gamma = (\gamma_1, \gamma_2, \cdots) \in l^2$, we have

$$\gamma(x_n - x_0) = \langle x_n - x_0, \gamma \rangle = \sum_{n=1}^{\infty} \gamma_n x_n = y_{n+1}.$$

But from definition, $\sum_{n=1}^{\infty} \| x_n \|^2 < \infty$, and consequently, $\lim_{n\to\infty} y_{n+1} = 0$. Moreover,

$$\lim_{n\to\infty}(\gamma(x_n - x_0)) = \lim_{n\to\infty}(\langle x_n - x_0, \gamma \rangle) = \lim_{n\to\infty} y_{n+1} = 0$$

. Hence $x_n \rightharpoonup x_0 = 0$.

Next, we define a mapping $G : X \to X$ by

$$Gx := \frac{1}{2}x,\ \forall x \in X$$

It is clear that $Fix(G) := \{0\} = p$. Since $X = l^2$ and $x_n \subset X$, we set here $x = x_n$ and converges weakly to 0. Clearly $\hat{F}ix(T) = Fix(T) = \{0\}$.

Now, setting $h(x) = \frac{2}{3} \| x \|^2,\ \nabla h(x) = \frac{4}{3} \| x \|$, we have

$$d_h(0, Gx) = h(0) - h(Gx) - \langle \nabla h(Gx), 0 - Gx \rangle$$
$$= 0 - \frac{1}{6} \| x \|^2 + \frac{1}{3} \| x \|^2 = \frac{1}{6} \| x \|^2,$$
$$d_h(0, x) = h(0) - h(x) - \langle \nabla h(x), 0 - x \rangle$$
$$= 0 - \frac{2}{3} \| x \|^2 + \frac{4}{3} \| x \|^2 = \frac{2}{3} \| x \|^2.$$

Thus

$$d_h(p, Gx) \le d_h(p, x),\ \forall x \in X.$$

Hence, G is a Bregman relatively nonexpansive which implies Bregman quasi-nonexpansive mapping since $Fix(G) = \hat{F}ix(G)$. If $x_n \subset K$ which is bounded, then G becomes a Bregman strongly nonexpansive mapping. The converse of the above statement is not true in general, i.e, Every Bregman quasi-nonexpansive mapping is not Bregman relatively nonexpansive and $Fix(T) \subset \hat{F}ix(T)$, (see Chan et al. (2011), Naraghirad and Yao (2013) for more information).

A mapping $g : K \times K \to R$ is called bifunctions so that the equilibrium problem with respect to $g : K \times K \to R$ introduced by Blum and Oettli (1994) is to find $p \in K$ such that

$$g(p,z) \ge 0 \quad \forall z \in K. \tag{7}$$

The set of solution of (7) is represented by $EP(K,g) = \{p \in K : g(p,z) \ge 0 \ \ \forall z \in K\}$.

To solve a problem of the form (7), certain conditions are imposed on the bifunctions $g : K \times K \to R$ as follows:

(A1) $g(x,x) = 0,\ \forall x \in k$,

(A2) $g : K \times K \to R$ is monotone

(A3) $\limsup_{t\downarrow 0} g((1-t)x + tz, y) \le g(x,y),\ \forall x,y,z \in K,$

(A4) The function $y \mapsto g(x,y)$ is convex and lower-semicontinuous.

The Resolvent of a bifunctions $g: K \times K \to R$ is the operator $\operatorname{Re} s_g^h : X \to 2^K$ defined by

$$\operatorname{Re} s_g^h(x) = \{p \in K : g(p,z) + \langle \nabla h(p) - \nabla h(x), z - p\rangle \geq 0, \ \forall \ z \in K\}. \quad (8)$$

This resolvent (8) is found to be Bregman quasi-nonexpansive (see Reich and Sabach, (2010), (2011) for more information)

It is a fact that the effective tool for finding common solution for the above nonlinear operators has been the iterative approximation methods. Several constructed iterative approximation methods seeking convergence abound in the literature (see Ofoedu et al. (2014) and the references therein). Basically, there are two known convergence which are weak and strong convergence denoted by $\rightharpoonup$ and $\rightarrow$ respectively. It is a fact that strong convergence is better than weak convergence. Hence, most published results only focused on the strong convergence of the formulated methods to the fixed point sets (see Shehu and Ogbuisi (2015), Zegeye (2014), Reich and Sabach (2010), Mainge (2008), Kohsaka and Takahashi (2005), Kohsaka and Takahashi, (2007), Ekuma-Okereke and Oladipo, Abkar, and Shekarbaigi (2019) and volume of others). In particular and for our purpose in this paper, Shehu and Ogbuisi (2015) studied the hybrid method of approximation common fixed points of Bregman strongly nonexpansive mappings and solutions of equilibrium problems as follows:

$$\begin{cases} x_0 \in K = K_0, \\ y_n^i = \nabla h^*(\alpha_n \nabla h(x_n) + (1-\alpha_n)\nabla h(T_i x_n)), \\ u_n^i = \operatorname{Re} s_{g_N}^h \circ \operatorname{Re} s_{g_{N-1}}^h \circ \ldots \circ \operatorname{Re} s_{g_2}^h \circ \operatorname{Re} s_{g_1}^h y_n^i, \\ K_{n+1} = \left\{ u \in K_n : \sup_{i \geq 1} d_h\left(u, u_n^i\right) \leq d_h\left(u, x_n\right)\right\}, \\ x_{n+1} = P_{K_{n+1}}^h(x_0), \quad n \geq 0. \end{cases}$$

where $P_{K_{n+1}}^h(x_0)$ is the Bregman projection mapping from X onto K, $\operatorname{Re} s_g^h$ is the resolvent operator in Banach space, $\{\alpha_n\}$ is a scalar sequence in (0,1) satisfying certain conditions. They proved that the sequence $\{x_n\}$ generated by their algorithms (9) converges strongly to a common solution of the problem $F = \bigcap_{i=1}^{\infty} Fix(T_i) \bigcap_{j=1}^{N} EP(g_j)$.

However, very few authors have paid attention to the possible speeding up of convergence of sequence of iterates generated by their constructed methods for different classes of Bregman nonexpansive-type nonlinear operators. Thus, a two-step iterative method known as inertial extrapolation algorithm which increases the rate of convergence was introduced and studied by Polyak (1964) and recently used in the works of Majee and Nahak (2018), Chidume et al. (2018), Dong et al. (2018), Ekuma-Okereke and Okoro (2020) which is defined as

$$u_{n+1} = u_n + \beta_n (u_n - u_{n-1}) \quad (10)$$

for all non-negative integers n, where $\beta_n \in (0,1)$. We note here that the two step approach is the fact that two initial and arbitrarily chosen values are used instead of the simple one value initialization.

Based on the above, Majee and Nahak (2018) introduced the Projection method in two ways: Parallel and Sequential respectively which is a combination of inertial extrapolation term and the well-known Mann iteration method for solving common point problem involving the fixed point of finite family of nonexpansive mappings and system of finite equilibrium problems in Hilbert space. Below are their algorithms:

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n^i = T_{r_n}^{g_i}(w_n),\ i = 1,2,...,N, \\ t_n = \dfrac{y_n^1 + y_n^2 + ... + y_n^N}{N}, \\ x_{n+1} = (1-\lambda_n)w_n + \lambda_n T_m^n T_{m-1}^n ... T_1^n t_n \quad n \geq 1, \end{cases} \tag{11}$$

and

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n^1 = T_{r_n}^{g_1}(w_n), \\ y_n^2 = T_{r_n}^{g_2}(y_n^1), \\ \vdots \\ y_n^N = T_{r_n}^{g_N}(y_n^{N-1}), \\ x_{n+1} = (1-\lambda_n)w_n + \lambda_n T_m^n T_{m-1}^n ... T_1^n y_n^N \quad n \geq 1, \end{cases}$$

(12

where T_r^f is the resolvent operator in Hilbert space, g is a bifunctions, T is a nonexpansive mapping, $\{\lambda_n\}$ is a scalar sequence in (0,1) satisfying certain conditions. They proved that the sequence $\{x_n\}$ generated by their algorithms (11) and (12) converges weakly to a common solution of the problem.

Our motivation for this study is the results of Majee and Nahak (2018). It is our aim to construct new iterative algorithms with inertial term and the hybrid iterative method involving the intersection of common fixed points of finite family of left Bregman strongly nonexpansive mappings and common element of the set of solutions of a finite family of equilibrium problems in a real reflexive Banach space. The specific objectives are to demonstrate: that the constructed algorithms find the fixed points of finite family of left Bregman strongly nonexpansive mappings, which solves equilibrium problems; that the sequence of iterates generated by our algorithms converge strongly to the Bregman Projection of the initial and arbitrarily chosen point onto the intersection of the sets. Furthermore, we demonstrate by the use of numerical software, that our algorithms are effective, implementable and converge faster with less computer time. The method of our proof and results obtained are well involved and significantly improves, extends many cited works in the literature.

2. Theoretical Framework

The following lemmas shall be used in the proof of our main results.

Lemma 2.1 (see Butnariu and Iusem (2000)). The function $h: X \to (-\infty,+\infty]$ is totally convex on bounded sets if and only if for any two sequences $\{x_n\}$ and $\{y_n\}$ in X such that either $\{x_n\}$ or $\{y_n\}$ is bounded, then

$$\lim_{n\to\infty} d_h(y_n, x_n) = 0 \Rightarrow \| y_n - x_n \| = 0$$

.

Lemma 2.2 (see Reich and Sabach (2011)). Let K be a non-void, closed and convex subsets of $\text{int}(dom\,h)$ and $G: K \to K$ be a left Bregman quasi nonexpansive mapping with respect to h. Then $Fix(G)$ is closed and convex.

Lemma 2.3. (see Martin-Marquez (2013)). Let $h: X \to (-\infty,+\infty]$ be a bounded, uniformly Fréchet differentiable and totally convex function on bounded subsets of X . Assume that ∇h^* is bounded subsets of $dom\,h^* = X^*$, and let K be a non-void subset of $\text{int}(dom\,h)$ and $\{G_i\}_{i=1}^M : K \to K$ be M left Bregman strongly-quasi nonexpansive mappings satisfying $\bigcap_{i=1}^{M} \hat{F}ix(G_i)$ is non-void, then G is left Bregman strongly-quasi nonexpansive mapping with $\hat{F}ix(G) = \bigcap_{i=1}^{M} \hat{F}ix(G_i)$.

Lemma 2.4 (see Bauschke et.al (2001)). Let X be a reflexive Banach space and let $h: X \to (-\infty, +\infty]$ be a continuous convex function which is strongly coercive. Then the following assertions are equivalent:

(i). h is bounded on bounded subsets and uniformly smooth on bounded subsets of X

(ii). h^* is Fr$\acute{e}$chet differentiable and ∇h^* is uniformly norm-to-norm continuous on bounded subsets of X^*.

(iii). $dom h^* = X^*$, h^* is strongly coercive and uniformly convex on bounded subsets of X^*.

Lemma 2.5 (see Butnariu and Resmerita (2006), Reich and Sabach (2010)). Let $h: X \to (-\infty, +\infty]$ be a $G\hat{a}teaux$ differentiable on $\text{int}(dom h)$ such that ∇h^* is bounded on bounded subsets of $dom h^*$. Let $x_0 \in X$ and $\{x_n\}$ is a sequence in X. If $\{d_h(x_0, x_n)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.

Lemma 2.6 (see Shehu and Ogbuisi (2015)). Let $h: X \to (-\infty, +\infty]$ be a Legendre function and K a non-void, closed and convex subset of X. If the bifunctions $g: K \times K \to R$ satisfies condition (A1)-(A4), then the following hold:

(i). $\text{Re}\, s_g^h$ is single valued

(ii). $Fix(\text{Re}\, s_g^h) = EP(K, g)$

(iii). $d_h(p, \text{Re}\, s_g^h(x)) + d_h(\text{Re}\, s_g^h(x), x) \leq d_h(p, x) \ \forall p \in Fix(\text{Re}\, s_g^h)$

(iv). $EP(K, g)$ is closed and convex.

Lemma 2.7 (see Naraghirad and Yao (2013)). Let $h: X \to (-\infty, +\infty]$ be a proper, lower semi-continuous and convex function, then $h^*: X^* \to (-\infty, +\infty]$ is a proper, $weak^*$ lower semi-continuous and convex function. Thus, for all $u \in X$ we have

$$d_h(u, \nabla h^* \textstyle\sum_{i=1}^{N} \alpha_i \nabla h(x_i)) \leq \sum_{i=1}^{N} \alpha_i d_h(u, x_i).$$

The Bregman Projection $u \in \text{int}(dom\ h)$ onto $K \subset dom h$, is the unique $u_0 \in K$ such that the mapping $P_K^h: \text{int}(dom h) \to K$ satisfy

$$d_h(u_0, u) = \min\{d_h(z, u): z \in K\}$$

and $P_K^h(u) = u_0$. The Bregman Projection mapping satisfy the following results:

Lemma 2.8 (see Chen et al. (2011)). Let K be non-void, closed, convex subsets of X. Let $h: X \to (-\infty, +\infty]$ be $G\dot{a}teaux$ differentiable and totally convex function and let $x \in X$, then

(i). $z = P_K^h(x)$ if and only if $\langle \nabla h(x) - \nabla h(z), y - z \rangle \leq 0, \ \forall y \in K$,

(ii). $d_h(y, P_K^h(x)) + d_h(P_K^h(x), x) \leq d_h(y, x) \ \ \forall y \in K.$

3. Main Results

In this section, we present the main results of this paper.

3.1. Sequential algorithm for finding common solution of fixed point and equilibrium problems

Theorem 3.1: Let K be a non-void, closed and convex subset of $\text{int}(dom h) \subset X$. Let $h: X \to R$ be a Legendre function which is bounded, uniformly Fr$\acute{e}$chet differentiable and totally convex on bounded subsets of a real reflexive Banach space X with $h^*: X^* \to R$ as its Fenchel conjugate. Let ∇h and ∇h^* be the gradient functions of $h: X \to R$ and

$h^*: X^* \to R$ respectively. Let $\{g_i\}_{i=1}^N : K \times K \to R$ be N-bifunctions which satisfy properties (A1)-(A4). Let $\{G_j\}_{j=1}^M : K \to K$ be M-finite family of Bregman strongly-quasi nonexpansive mappings induced by a convex function $h: X \to R$ such that $\hat{F}ix(\{G_j\}_{j=1}^M) = Fix(\{G_j\}_{j=1}^M)$. Assume that $F = \bigcap_{i=1}^N EP(g_i) \cap \left(\bigcap_{j=1}^M Fix(G_j)\right)$ is non-void. Set $x_0, x_1 \in K$. Define a sequence $\{x_n\}$ by the following manner:

$$\begin{cases} x_0, x_1 \in K \;\; Chosen\ arbitrarily, \\ K_1 = K, \\ z_n = \nabla h^*\left(\nabla h(x_n) + \alpha_n \nabla h(x_n - x_{n-1})\right), \\ y_n = \nabla h^*\left(b_n \nabla h(z_n) + (1-b_n)\nabla h(Gz_n)\right), \\ w_n^1 = \operatorname{Re} s_{g_1}^h(y_n), \\ w_n^2 = \operatorname{Re} s_{g_2}^h(w_n^1) \\ \vdots \\ w_n^N = \operatorname{Re} s_{g_N}^h(w_n^{N-1}), \\ K_{n+1} = \{u \in K_n : d_h(u, w_n^N) \le d_h(u, z_n)\}, \\ x_{n+1} = P_{K_{n+1}}^h(x_0), \quad n \ge 1. \end{cases} \tag{13}$$

where $G = G_M \circ G_{M-1} \circ \cdots \circ G_1,\ \theta_n^j = G_j \circ G_{j-1} \circ \cdots \circ G_1 z_n,\ \forall j \ge 1$ and $P_{K_{n+1}}^h$ is the Bregman projection of $\operatorname{int}(dom\, h)$ onto K. Suppose $\{\alpha_n\} \subset (a,b) \subset (0,1)$, $\{b_n\} \subset (0,1)$ are scalar sequences such that $\liminf_{n\to\infty}(1-b_n) > 0$, then the sequences $\{x_n\}, \{z_n\}$ converges strongly to a common element of F.

Proof: We shall demonstrate the analytical proof of the theorem in the following steps.

Step 1: The sequence $\{x_n\}$ defined by (3.1) is well-defined for each $n \ge 1$.

We first show that $F = \bigcap_{i=1}^N EP(g_i) \cap \left(\bigcap_{j=1}^M Fix(G_j)\right)$ is closed and convex. From Lemmas 2.2, 2.3 and 2.6 we have that $F = \bigcap_{i=1}^N EP(g_i) \cap \left(\bigcap_{j=1}^M Fix(G_j)\right)$ is closed and convex. Next is to show that that K_n is closed and convex for each $n \ge 1$. To see this, from definition of K_n, we get that K_n is closed. Moreover, since $d_h(u, w_n^N) \le d_h(u, z_n)$ is equivalent to

$$\left\langle \nabla h(z_n) - \nabla h(w_n^N), u \right\rangle$$
$$+ \left\langle \nabla h(z_n) - \nabla h(w_n^N), w_n^N - z_n \right\rangle \le h(w_n^N) - h(z_n)$$

it follows that K_n is a half space and hence convex for each $n \ge 1$.

In addition, we show that $F \subset K_n$ for each $n \ge 1$. It is clear from the initial assumption that $F \subset K_1 = K$. Now suppose that $F \subset K_n$ for some numbers $n > 1$, then for $p \in F$, and using (P1) together with Lemma 2.6, we obtain

$$\begin{aligned} d_h(p, w_n^1) &= d_h(p, \operatorname{Re} s_{g_1}^h(y_n)) \\ &\le d_h(p, y_n). \end{aligned} \tag{14}$$

Furthermore,

$$\begin{aligned} & d_h(p, y_n) = \\ & d_h(p, \nabla h^*(b_n \nabla h(z_n) + (1-b_n)\nabla h(Gz_n))) \\ & \le b_n d_h(p, z_n) + (1-b_n)\, d_h(p, Gz_n) \\ & = b_n d_h(p, z_n) + (1-b_n)\, d_h(p, \theta_n^M) \\ & \le b_n d_h(p, z_n) + (1-b_n) d_h(p, \theta_n^{M-1}) \\ & \quad \vdots \\ & \le b_n d_h(p, z_n) + (1-b_n) d_h(p, G_1 z_n) \\ & \le b_n d_h(p, z_n) + (1-b_n) d_h(p, z_n) \\ & = d_h(p, z_n). \end{aligned} \tag{15}$$

Similarly,

$$\begin{aligned} d_h(p, w_n^2) &= d_h(p, \operatorname{Re} s_{g_2}^h(w_n^1)) \\ &\le d_h(p, w_n^1) \\ &\le d_h(p, y_n), \\ &\vdots \\ d_h(p, w_n^N) &= d_h(p, \operatorname{Re} s_{g_N}^h(w_n^{N-1})) \end{aligned}$$

$$\le d_h(p, w_n^{N-1})$$
$$\le d_h(p, y_n),$$

Using (15) we have

$d_h(p, w_n^N) \le d_h(p, z_n).$ (16)

So $p \in K_{n+1}$ and $K_{n+1} \subset K_n$. This implies by set induction that $F \subset K_n$. Thus, $\{x_n\}$ is well defined for all $n \ge 1$. This completes the proof of step 1.

Step 2: The following estimates holds

(i) $\lim_{n\to\infty} d_h(x_{n+1}, x_n) = 0 \Rightarrow \lim_{n\to\infty} \| x_{n+1} - x_n \| = 0,$

(ii) $\lim_{n\to\infty} \| x_n - z_n \| = 0 \Rightarrow \lim_{n\to\infty} d_h(x_n, z_n) = 0$

(iii) $\lim_{n\to\infty} d_h(x_{n+1}, w_n^N) = 0 \Rightarrow \lim_{n\to\infty} \| x_{n+1} - w_n^N \| = 0,$

(iv) $\lim_{n\to\infty} \| x_n - y_n \| = 0 \Rightarrow \lim_{n\to\infty} d_h(x_n, y_n) = 0$

(v) $\lim_{n\to\infty} d_h(Gz_n, z_n) = 0 \Rightarrow \lim_{n\to\infty} \| z_n - Gz_n \| = 0,$

To demonstrate (i), notice from algorithm (13) above that $x_n = P_{K_n}^h(x_0)$ and $x_{n+1} = P_{K_{n+1}}^h(x_0) \in K_{n+1} \subset K_n$. Thus, we obtain from Lemma 2.8 that

$d_h(x_n, x_0) \le d_h(x_{n+1}, x_0) - d_h(x_{n+1}, x_n)$

$d_h(x_n, x_0) \le d_h(x_{n+1}, x_0).$ (17)

Again, we get from Lemma 2.8 that

$d_h(x_n, x_0) =$

$d_h(P_{K_n}^h(x_0), x_0) \le d_h(p, x_0) - d_h(p, P_{K_n}^h(x_0))$

$\le d_h(p, x_0) \quad \forall n \ge 1,\ p \in F.$

This implies that

$$d_h(x_n, x_0) \le d_h(p, x_0). \qquad (18)$$

This shows that $\{d_h(x_n, x_0)\}$ is bounded and monotone nondecreasing sequence of real numbers. From Lemma 2.5, we get that $\{x_n\}, \{y_n\}, \{z_n\}, \{w_n^N\}$ are bounded. Combining (17) and (18), we get that $\lim_{n\to\infty} d_h(x_n, x_0)$ exist. Now wlog, let

$$\lim_{n\to\infty} d_h(x_n, x_0) = l \qquad (19)$$

In addition to (19) and Lemma 2.8, we get that for any positive integer, μ,

$$d_h(x_{n+\mu}, x_n) = d_h(x_{n+\mu}, P_{K_n}^h(x_0))$$

$\le d_h(x_{n+\mu}, x_0) - D_f(x_n, x_0) \to 0$ as $n \to \infty$.

So that

$$\lim_{n\to\infty} d_h(x_{n+\mu}, x_n) = 0.$$

In particular,

$$\lim_{n\to\infty} d_h(x_{n+1}, x_n) = 0. \qquad (20)$$

By Lemma 2.1, (19) implies that

$$\lim_{n\to\infty} \| x_{n+1} - x_n \| = 0. \qquad (21)$$

This establishes identity (i).

From (21) we conclude that the sequence $\{x_n\}$ is a Cauchy sequence in K. Using the fact that X is complete and K is closed, we get that $x_n \to z_0 \in K$ as $n \to \infty$. Now, from the uniform continuity of ∇h, we have

$$\lim_{n\to\infty} \| \nabla h(x_{n+1}) - \nabla h(x_n) \| = 0. \qquad (22)$$

Furthermore, from the definition of z_n, and together with (22) we have that

$$\| \nabla h(x_n) - \nabla h(z_n) \| =$$
$$\| \nabla h(x_n) - \nabla h(x_n) - \alpha_n \nabla h(x_n - x_{n-1}) \|$$
$$= \| \alpha_n \nabla h(x_{n-1} - x_n) \|$$
$$\le \| \nabla h(x_{n-1} - x_n) \| \to 0 \text{ as } n \to \infty.$$

This implies that

$$\lim_{n\to\infty} \| \nabla h(x_n) - \nabla h(z_n) \| = 0. \qquad (23)$$

By Lemma 2.4, we obtain that

$$\lim_{n\to\infty} \| x_n - z_n \| = 0. \qquad (24)$$

This establishes identity (ii) and shows that $z_n \to z_0$ as $n \to \infty$.

Moreso, since $\{ z_n \}$ is bounded and using (P6) and (24), we have that

$$\lim_{n\to\infty} d_h(x_n, z_n) = 0. \qquad (25)$$

In addition, since $x_{n+1} \in K_{n+1} \subset K_n$, we have from the definition of the half space that

$$d_h(x_{n+1}, w_n^N) \le d_h(x_{n+1}, z_n). \quad (26)$$

Moreover, using (P4), (20), (23) and (25), we get

$$\begin{aligned} 0 \le d_h(x_{n+1}, z_n) &= d_h(x_{n+1}, x_n) + d_h(x_n, z_n) \\ &+ \langle \nabla h(x_n) - \nabla h(z_n), x_{n+1} - x_n \rangle \\ &\le d_h(x_{n+1}, x_n) + d_h(x_n, z_n) \\ &+ \| \nabla h(x_n) - \nabla h(z_n) \| . \| x_{n+1} - x_n \| \to 0 \end{aligned}$$

This demonstrate that

$$\lim_{n\to\infty} d_h(x_{n+1}, z_n) = 0. \quad (27)$$

This implies that

$$\lim_{n\to\infty} d_h(x_{n+1}, w_n^N) = 0. \quad (28)$$

Thus, by Lemma 2.1, (27) and (28) implies that

$$\lim_{n\to\infty} \| x_{n+1} - z_n \| = 0$$

and

$$\lim_{n\to\infty} \| x_{n+1} - w_n^N \| = 0. \quad (29)$$

This establishes identity (iii).

Furthermore,

$$\begin{aligned} d_h(y_n, w_n^1) &= d_h(y_n, \operatorname{Res}_{g_1}^h y_n) \\ &\le d_h(p, \operatorname{Res}_{g_1}^h y_n) - d_h(p, y_n) \\ &\le d_h(p, y_n) - d_h(p, y_n) \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

$$\begin{aligned} d_h(y_n, w_n^2) &= d_h(y_n, \operatorname{Res}_{g_2}^h (w_n^1)) \\ &\le d_h(p, \operatorname{Res}_{g_2}^h (w_n^1)) - d_h(p, y_n) \\ &\le d_h(p, w_n^1) - d_h(p, y_n) \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Continuing this process, we get

$$d_h(y_n, w_n^N) = d_h(y_n, \operatorname{Res}_{g_N}^h (w_n^{N-1})) \le d_h(p, \operatorname{Res}_{g_N}^h (w_n^{N-1})) - d_h(p, y_n)$$

$$\le d_h(p, w_n^{N-1}) - d_h(p, y_n) \to 0 \text{ as } n \to \infty. \quad (30)$$

This implies that

$$\lim_{n\to\infty} d_h(y_n, w_n^N) = 0. \quad (31)$$

By Lemma 2.1, (31) implies that

$$\lim_{n\to\infty} \| w_n^N - y_n \| = 0. \quad (32)$$

Hence in general, we arrive at

$$\lim_{n\to\infty} d_h(y_n, w_n^i) = 0, \ \forall i = 1,2,\ldots, N. \quad (33)$$

By Lemma 2.1, (33) implies that

$$\lim_{n\to\infty} \| w_n^i - y_n \| = 0, \ \forall\, i = 1,2,\ldots, N. \quad (34)$$

Now, from the uniform continuity of ∇h, (32) becomes

$$\lim_{n\to\infty} \| \nabla h(w_n^i) - \nabla h(y_n) \| = 0. \quad (35)$$

Thus adding (29) and (32) gives

$$\lim_{n\to\infty} \| x_{n+1} - y_n \| = 0. \quad (36)$$

So that

$$\lim_{n\to\infty} \| x_n - y_n \| = 0. \quad (37)$$

This establishes (vi).

Using (37) we deduce from (P6) that $d_h(x_n, y_n) \to 0 \ as \ n \to \infty$. By (P4) and the uniform continuity of ∇h on bounded sets, we get

$$\begin{aligned} d_h(z_n, y_n) &= d_h(z_n, x_n) + d_h(x_n, y_n) \\ &+ \langle \nabla h(y_n) - \nabla h(x_n), x_n - z_n \rangle \\ &\le d_h(z_n, x_n) + d_h(x_n, y_n) + \| x_n - z_n \| \\ &\times \| \nabla h(x_n) - \nabla h(y_n) \| \to 0 \ as \ n \to \infty \end{aligned} \quad (38)$$

Again, by (P4) and the uniform continuity of ∇h on bounded sets, we get

$$d_h(u, z_n) - d_h(u, y_n) =$$

$$\langle u - y_n, \nabla h(y_n) - \nabla h(z_n) \rangle - d_h(z_n, y_n)$$

$$\le \| u - y_n \| \| \nabla h(z_n) - \nabla h(y_n) \| - d_h(z_n, y_n) \to 0 \ as \ n \to \infty.$$

Thus

$$d_h(u, z_n) - d_h(u, y_n) \to 0, \ as\, n \to \infty. \quad (39)$$

Furthermore, since $y_n = \nabla h^* (b_n \nabla h(z_n) + (1 - b_n) \nabla h(Gz_n))$, observe that

$$\begin{aligned} d_h(u, z_n) - d_h(u, Gz_n) &= d_h(u, z_n) - d_h(u, y_n) \\ &+ d_h(u, y_n) - d_h(u, Gz_n) \\ &= d_h(u, z_n) - d_h(u, y_n) + \\ &d_h(u, \nabla h^*(b_n \nabla h(z_n) + (1 - b_n) \nabla h(Gz_n))) - d_h(u, Gz_n) \end{aligned}$$

$$\le d_h(u, z_n) - d_h(u, y_n) + b_n d_h(u, z_n) +$$
$$d_h(u, z_n) - d_h(u, Gz_n) \le$$

$(1-b_n)d_h(u,Gz_n)-d_h(u,Gz_n)$
$=d_h(u,z_n)-d_h(u,y_n)+b_n[d_h(u,z_n)-d_h(u,Gz_n)]$

$\frac{1}{(1-b_n)}(d_h(u,z_n)-d_h(u,y_n))$.

Using (39) and the fact that $\liminf_{n\to\infty}(1-b_n)>0$ we get

$d_h(u,z_n)-d_h(u,Gz_n)\to 0$, as $n\to\infty$.

Since G_j for each $j=1,2,...,M$, and hence $G=G_j\circ G_{j-1}\circ....\circ G_1$ are left Bregman strongly nonexpansive mappings, it follows that

$d_h(Gz_n,z_n)\to 0$, as $n\to\infty$. (40)

This implies by Lemma 2.1 that

$$\lim_{n\to\infty}\|z_n-Gz_n\|=0. \quad (41)$$

This establishes (v). Hence this completes the proof of step 2.

Step 3: We demonstrate that $z_0\in\left(\bigcap_{i=1}^N EP(g_i)\right)\bigcap\left(\bigcap_{j=1}^M Fix(G_j)\right)$

First, we demonstrate that $z_0\in\bigcap_{j=1}^M Fix(G_j)$.

Since $z_n\to z_0$ and the fact that $G=G_j\circ G_{j-1}\circ....\circ G_1$ is closed under composition together with $\hat{F}ix(G)=Fix(G)=\bigcap_{j=1}^M Fix(G_j)$, we get that $z_0\in\bigcap_{j=1}^M Fix(G_J)$.

Next, we demonstrate that $z_0\in\bigcap_{i=1}^N EP(g_i)$.

By definition, we have for each $i=1,2,...,N$, that

$$g_i(w_n^i,y)+\langle\nabla h(w_n^i)-\nabla h(y_n),y-w_n^i\rangle\ge 0,\ \forall\, y\in K$$
$$\langle\nabla h(w_n^i)-\nabla h(y_n),y-w_n^i\rangle\ge g_i(y,w_n^i),\ \forall\, y\in K,$$
$$\|\nabla h(w_n^i)-\nabla h(y_n)\|\|y-w_n^i\|$$
$$\ge\langle\nabla h(w_n^i)-\nabla h(y_n),y-w_n^i\rangle-g_i(y,w_n^i)$$

This implies that

$\|\nabla h(w_n^i)-\nabla h(y_n)\|\|y-w_n^i\|\ge g_i(y,w_n^i)$ $\forall\, y\in K$. (42)

Since $g_i(y,w_n^i)\ \forall\, y\in K,\forall i=1,2,....,N$ is convex and lower semicontinuous, $w_n^i\to z_0\ \forall i=1,2,....,N$, we get from (35) that

$g_i(y,z_0)\le 0\ \ \forall\, y\in K$. (43)

We set $\lambda\in(0,1)$ and $w_\lambda=\lambda y+(1-\lambda)z_0$, so that $w_\lambda\in K$.

This demonstrates that $g_i(w_\lambda,z_0)\le 0$ $\forall\, y\in K$. Using this, together with (A1) and (A4), we get

$$0=g_i(w_\lambda,w_\lambda)=g_i(w_\lambda,\lambda y+(1-\lambda)z_0)$$
$$\le\lambda g_i(w_\lambda,y)+(1-\lambda)g_i(w_\lambda,z_0)\le\lambda g_i(w_\lambda,y)$$

This implies that

$g_i(w_\lambda,y)\ge 0$. (44)

By (A3), we get that

$$g_i(z_0,y)\ge 0,\ y\in K,\ i=1,2,...,N.$$

We conclude that $z_0\in\bigcap_{i=1}^N EP(g_i)$.

Thus, $z_0\in\left(\bigcap_{i=1}^N EP(g_i)\right)\bigcap\left(\bigcap_{j=1}^M Fix(G_j)\right)$.

This completes the proof of step 3

Step 4: We demonstrate that $x_n\to z_0=P_F^h(x_0)$. Since $x_n=P_{K_n}^h(x_0)$ and from step 1, $F\subset K_n$ so that from Lemma 2.8, we have

$$d_h(x_0,x_{n+1})+d_h(x_{n+1},P_F^h(x_0))\le d_h(x_0,P_F^h(x_0)) \quad (45)$$

Since $x_n\to z_0$ and by taking limit on both sides of (45), we get

$$d_h(x_0,z_0)+d_h(z_0,P_F^h(x_0))\le d_h(x_0,P_F^h(x_0)).$$

This implies

$d_h(x_0,z_0)\le d_h(x_0,P_F^h(x_0))$. (46)

On the other hand, we get using Lemma 2.8 that

$$d_h(x_0,P_F^h(x_0))+d_h(P_F^h(x_0),z_0)\le d_h(x_0,z_0).$$

This implies

$d_h(x_0,P_F^h(x_0))\le d_h(x_0,z_0)$ (47)

By combining (46) and (47), we have

$$d_h(x_0, P_F^h(x_0)) = d_h(x_0, z_0) \quad (48)$$

By the uniqueness property of $P_F^h(x_0)$, we conclude that $x_n \to z_0 = P_F^f(x_0)$. Consequently, $z_n \to z_0 = P_F^f(x_0)$. Thus the proof of step 4 is complete. Therefore, this completes the proof of theorem 3.1.

3.2. Parallel algorithm for finding common solution of fixed point and equilibrium problems

Theorem 3.2: Let K be a non-void, closed and convex subset of $\operatorname{int}(dom h) \subset X$. Let $h: X \to R$ be a Legendre function which is bounded, uniformly Fr$\acute{e}$chet differentiable and totally convex on bounded subsets of a real reflexive Banach space X with $h^*: X^* \to R$ as its Fenchel conjugate. Let ∇h and ∇h^* be the gradient functions of $h: X \to R$ and $h^*: X^* \to R$ respectively. Let $\{g_i\}_{i=1}^N : K \times K \to R$ be N-bifunctions which satisfy properties (A1)-(A4). Let $\{G_j\}_{j=1}^M : K \to K$ be M-finite family of Bregman strongly-quasi nonexpansive mappings induced by a convex function $h: X \to R$ such that $\hat{F}ix(\{G_j\}_{j=1}^M) = Fix(\{G_j\}_{j=1}^M)$. Assume that $F = \bigcap_{i=1}^N EP(g_i) \cap \left(\bigcap_{j=1}^M Fix(G_j)\right)$ is non-void. Set $x_0, x_1 \in K$. Define a sequence $\{x_n\}$ by the following manner:

$$\begin{cases} x_0, x_1 \in K, \text{ Chosen arbitrarily}, \\ K_1 = K, \\ z_n = \nabla h^*\left(\nabla h(x_n) + \alpha_n \nabla h(x_n - x_{n-1})\right), \\ y_n = \nabla h^*\left(b_n \nabla h(z_n) + (1-b_n)\nabla h(Gz_n)\right), \\ w_n^i = \operatorname{Re} s_{g_i}^h(y_n), \ i = 1,2,\dots, N, \\ t_n = \nabla h^*\left(\sum_{i=1}^N \frac{1}{N} \nabla h(w_n^i)\right), \\ K_{n+1} = \{u \in K_n : d_h(u, t_n) \le d_h(u, z_n)\}, \\ x_{n+1} = P_{K_{n+1}}^h(x_0), \quad n \ge 1. \end{cases} \quad (49)$$

where $G = G_M \circ G_{M-1} \circ \cdots \circ G_1,\ \theta_n^j = G_j \circ G_{j-1} \circ \cdots \circ G_1 z_n, \forall j \ge 1$, $P_{K_{n+1}}^h$ is the Bregman projection of $\operatorname{int}(dom h)$ onto K. Suppose $\{\alpha_n\} \subset (a,b) \subset (0,1)$, $\{b_n\} \subset (0,1)$ are scalar sequences $\liminf_{n\to\infty}(1-b_n) > 0$, then the sequences $\{x_n\}, \{z_n\}$ converges strongly to a common element of F.

Proof: Following the same method of proof of Theorem 3.1 and Lemma 2.7, then the conclusion holds.

4. Application to convex feasibility problem

We let $C_1, C_2, \dots C_M$ be a nonempty closed convex sets in reflexive real Banach space such that the finite intersection of these sets is not empty. The convex feasibility problem (CFP) is to find a point say $z \in C$. The Bregman Projection onto the i^{th} constraint C_i with respect to a Legendre function h is denoted by $P_{C_i}^h$.

It is easy to compute that $Fix(P_{C_i}^h) = C_i, \ \forall i = 1, 2, \dots, M$. We this fact, the fixed problem for finite family of Bregman nonexpansive-type mappings becomes solutions of finite family of convex feasibility problems. If in addition, the Legendre function is necessarily uniformly Fr$\acute{e}$chet differentiable and totally convex on bounded subsets of a real reflexive Banach space X, we get that $P^h_{C_i}$ is in particular a Bregman strongly nonexpansive mapping and $Fix(P^h_{C_i}) = \hat{F}ix(P^h_{C_i})$, (see Reich and Sabach (2010)). With these facts $G_i = P^h_{C_i}$ for all $i = 1, 2, ..., M$. Thus, using Theorem 3.1 we state a strong convergence results for common solution of convex feasibility and equilibrium problems as follows:

Theorem 4.1: Let K be a non-void, closed and convex subset of $\text{int}(dom h) \subset X$. Let $h : X \to R$ be a Legendre function which is bounded, uniformly Fr$\acute{e}$chet differentiable and totally convex on bounded subsets of a real reflexive Banach space X with $h^* : X^* \to R$ as its Fenchel conjugate. Let ∇h and ∇h^* be the gradient functions of $h : X \to R$ and $h^* : X^* \to R$ respectively. Let $\{g_i\}_{i=1}^N : K \times K \to R$ be N-bifunctions which satisfy properties (A1)-(A4). Let $G_j = P^h_{C_j}$ such that $\hat{F}ix(\{P^h_{C_j}\}_{j=1}^M) = Fix(\{P^h_{C_j}\}_{j=1}^M)$. Assume that $F = \bigcap_{i=1}^N EP(g_i) \bigcap \left(\bigcap_{j=1}^M Fix(P^h_{C_j})\right)$ is non-void. Set $x_0, x_1 \in K$. Define a sequence $\{x_n\}$ by the following manner:

$$\begin{cases} x_0, x_1 \in K \ \ Chosen\ arbitrarily, \\ K_1 = K, \\ z_n = \nabla h^*\left(\nabla h(x_n) + \alpha_n \nabla h(x_n - x_{n-1})\right), \\ y_n = \nabla h^*\left(b_n \nabla h(z_n) + (1-b_n)\nabla h(Gz_n)\right), \\ w_n^1 = \text{Re}\, s^h_{g_1}(y_n), \\ w_n^2 = \text{Re}\, s^h_{g_2}(w_n^1) \\ \vdots \\ w_n^N = \text{Re}\, s^h_{g_N}(w_n^{N-1}), \\ K_{n+1} = \{u \in K_n : d_h(u, w_n^N) \le d_h(u, z_n)\}, \\ x_{n+1} = P^h_{K_{n+1}}(x_0), \quad n \ge 1. \end{cases} \tag{50}$$

where $G = G_M \circ G_{M-1} \circ \cdots \circ G_1$, $\theta_n^j = G_j \circ G_{j-1} \circ \cdots \circ G_1$, for $G_i = P^h_{C_i}$, and $P^h_{K_{n+1}}$ is the Bregman projection of $\text{int}(dom h)$ onto K. Suppose $\{\alpha_n\} \subset (a,b) \subset (0,1)$, $\{b_n\} \subset (0,1)$ are scalar sequences such that $\liminf_{n\to\infty}(1-b_n) > 0$, then the sequences $\{x_n\}, \{z_n\}$ converges strongly to a common element of F.

Remark: Following similar approach above, Theorem 3.2 is also applied in common solution of convex feasibility and equilibrium problems.

5. Numerical Example:

We present a numerical example to justify our theoretical assertions and geometric properties made in section 3 of this paper. We remark that our codes were written in Python and run on PC with intel(R) Core(TM)2 Duo CPU @ 3.10 GHz processor.

5.1. Here is a simple illustration that provides the data in Tables 1 and 2 respectively, and Figure 1

Let $X = R,\ K = [0,1]$. Also consider $M = N = 10$. Consider the convex function $h: K \to R$ defined by $h(x) = (2/3)x^2$, such that $\nabla h(x) = (4/3)x$. The Fenchel conjugate function $h^*: K \to R$ using $h^*(x^*) = \sup\{\langle x^*, x\rangle - h(x): x \in X\}$, is $h^*(u) = \frac{3}{8}u^2$. So that $\nabla h^*(u) = \frac{3}{4}u,\ \forall u \in X$. Next, we define our mapping $G_j: K \to K,\ j = 1,2,...,M$, by

$$G_j x := \frac{j}{J+1}x,\ \forall x \in K,\ \ j = 1,2,...,M$$

It is clear that $\bigcap_{j=1}^{10} Fix(G_j) := \{0\} = p$.
Now if we set $G = G_{10} \circ G_9 \circ G_8 \circ \cdots \circ G_1$, so that $Gx = G_{10} \circ G_9 \circ G_8 \circ \cdots \circ G_1 x = \frac{x}{11}$ and $Fix(G) := \{0\}$. Hence $Fix(G) := \bigcap_{j=1}^{10} Fix(G_j) := \{0\}$. Moreover, for any sequence $x_n \subset K$ which is bounded, then G_j, for each $j \in \{1, 2, \cdots M\}$ and thus G are clearly Bregman strongly nonexpansive mappings. Moreover, using the fact that G is Bregman strongly nonexpansive, we have that $\hat{F}ix(G) := Fix(G) := \bigcap_{j=1}^{10} Fix(G_j) := \{0\} = p.$

Furthermore, we define the bifunctions $g_i: K \times K \to R$ for $i = 1,2...,N$ by

$$g_i(u,z) := i(2z^2 + uz - 3u^2).$$

There exist $u \in K$ such that

$$g_i(u,z) + \langle \nabla h(u) - \nabla h(y), z - u\rangle \geq 0,\ \ \forall z \in K,$$

$$i(2z^2 + uz - 3u^2) + \langle \frac{4}{3}u - \frac{4}{3}y, z - u\rangle \geq 0, z \in K,$$

which is equivalent to

$$2iz^2 + \left(iu + \frac{4}{3}u - \frac{4}{3}y\right)z - 3iu^2 - \frac{4}{3}u^2 + \frac{4}{3}yu \ \geq 0,\ z \in K.$$

Set

$$R(z) := 2iz^2 + \left(iu + \frac{4}{3}u - \frac{4}{3}y\right)z - 3iu^2 - \frac{4}{3}u^2 + \frac{4}{3}yu.$$

This function is a quadratic function with respect to z. Now using the discriminant of R, we get

$$D := \frac{1}{9}(15iu + 4u - 4y)^2.$$

Since $R(z) \geq 0\ \forall z \in K$ and since it has at most one solution in R, we get that

$$D := \frac{1}{9}(15iu + 4u - 4y)^2 \leq 0.$$

Clearly equality holds and solving for u, we get

$$u := \frac{4}{15i + 4}y.$$

This implies that

$$\operatorname{Re} s_{g_i}^h(y) := \frac{4}{15i + 4}y.$$

We assume for our purpose that

$$\alpha_n = \frac{1}{(n+10)^2},\ \ b_n = \frac{n+1}{3n}.$$

Using the above, our algorithms, tables and graphs of (13) and (49) respectively becomes:

$$\begin{cases} z_n = x_n + \alpha_n\left(x_n - x_{n-1}\right), \\ y_n = \dfrac{1}{33}\left(\dfrac{13n+10}{n}\right)z_n, \\ w_n^1 = \dfrac{4}{19}y_n, \\ w_n^2 = \dfrac{2}{17}w_n^1, \\ \vdots \\ w_n^{10} = \dfrac{2}{77}w_n^9, \\ K_{n+1} := \left\{u \in K_n : u \leq -\dfrac{3}{2}w_n^{10} + \dfrac{1}{2}z_n\right\} \\ x_{n+1} = P_{K_{n+1}}^h(x_0) = u,\ n \geq 1. \end{cases}$$

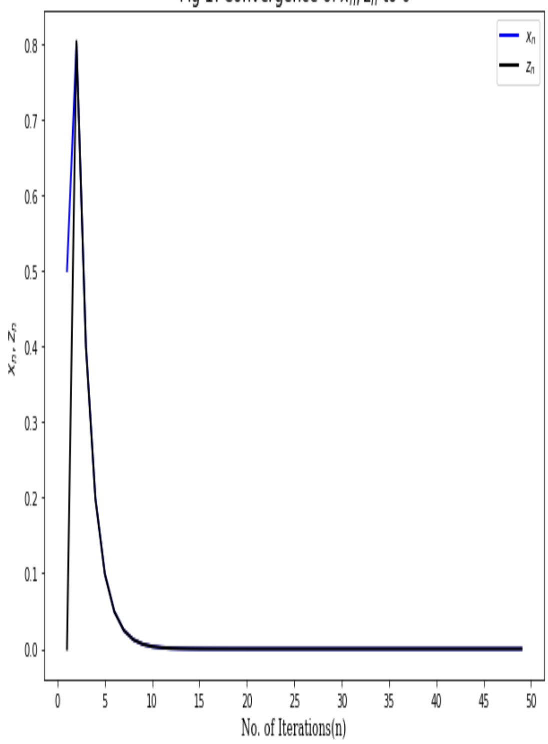
$$\begin{cases} z_n = x_n + \alpha_n\left(x_n - x_{n-1}\right), \\ y_n = \dfrac{1}{33}\left(\dfrac{13n+10}{n}\right)z_n, \\ w_n^i = \operatorname{Re}s_{g_i}^h(y_n) = \dfrac{4}{15i+4}y_n, i = 1,2,...,10, \\ t_n = \dfrac{\mathrm{w}_\mathrm{n}^1 + \mathrm{w}_\mathrm{n}^2 + \cdots + \mathrm{w}_\mathrm{n}^{10}}{10}, \\ K_{n+1} := \left\{u \in K_n : u \leq -\dfrac{3}{2}t_n + \dfrac{1}{2}z_n\right\} \\ x_{n+1} = P_{K_{n+1}}^h(x_0) = u,\ n \geq 1. \end{cases}$$

Itera[i]	x[i]	z[i]
0	0.500000	0.000000
1	0.800000	0.802479
2	0.401240	0.398471
3	0.199235	0.198040
4	0.099020	0.098509
5	0.049254	0.049033
6	0.024517	0.024420
7	0.012210	0.012167
8	0.006084	0.006065
9	0.003032	0.003024
10	0.001512	0.001508
11	0.000754	0.000752
12	0.000376	0.000375
13	0.000188	0.000187
14	0.000094	0.000094
15	0.000047	0.000047
16	0.000023	0.000023
17	0.000012	0.000012
18	0.000006	0.000006
19	0.000003	0.000003
20	0.000001	0.000001
21	0.000001	0.000001
22	0.000000	0.000000
23	0.000000	0.000000
24	0.000000	0.000000
25	0.000000	0.000000
26	0.000000	0.000000
27	0.000000	0.000000
28	0.000000	0.000000
29	0.000000	0.000000
30	0.000000	0.000000
31	0.000000	0.000000
32	0.000000	0.000000
33	0.000000	0.000000
34	0.000000	0.000000
35	0.000000	0.000000
36	0.000000	0.000000
37	0.000000	0.000000
38	0.000000	0.000000
39	0.000000	0.000000
40	0.000000	0.000000
41	0.000000	0.000000
42	0.000000	0.000000
43	0.000000	0.000000
44	0.000000	0.000000
45	0.000000	0.000000
46	0.000000	0.000000
47	0.000000	0.000000
48	0.000000	0.000000
49	0.000000	0.000000

Table 1: Computational Results of Sequential Algorithm of (13) with initialization $x_0 = 0.5,\ x_1 = 0.8$ for 50 iterations for all $n \geq 1$.

Key: itera[i] is the number of iterations, x[i] and z[i] are the values of sequences x_n and z_n respectively.

Fig 1: Convergence of x_n, z_n to 0



Itera[i]	x[i]	z[i]
0	0.500000	0.000000
1	0.800000	0.802479
2	0.343419	0.340248
3	0.150938	0.149799
4	0.067235	0.066808
5	0.030160	0.029995
6	0.013588	0.013523
7	0.006140	0.006115
8	0.002781	0.002771
9	0.001262	0.001257
10	0.000573	0.000571
11	0.000261	0.000260
12	0.000119	0.000118
13	0.000054	0.000054
14	0.000025	0.000025
15	0.000011	0.000011
16	0.000005	0.000005
17	0.000002	0.000002
18	0.000001	0.000001
19	0.000000	0.000000
20	0.000000	0.000000
21	0.000000	0.000000
22	0.000000	0.000000
23	0.000000	0.000000
24	0.000000	0.000000
25	0.000000	0.000000
26	0.000000	0.000000
27	0.000000	0.000000
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35	0.000000	0.000000
36	0.000000	0.000000
37	0.000000	0.000000
38	0.000000	0.000000
39	0.000000	0.000000
40	0.000000	0.000000
41	0.000000	0.000000
42	0.000000	0.000000
43	0.000000	0.000000
44	0.000000	0.000000
45	0.000000	0.000000
46	0.000000	0.000000
47	0.000000	0.000000
48	0.000000	0.000000
49	0.000000	0.000000

Table 2: Computational Results of Parallel Algorithm of (49) with initialization $x_0 = 0.5,\ x_1 = 0.8$ for 50 iterations for all $n \geq 1$.

Key: itera[i] is the number of iterations, x[i] and z[i] are the values of sequences x_n and z_n respectively.

5. Conclusion:

In this paper, we formulated two new hybrid iterative algorithms (Sequential and Parallel) with an inertial extrapolation component that solves a common solution problem of finite family of Bregman strongly-quasi-nonexpansive self-mappings and finite system of equilibrium in a reflexive and (real) Banach space. From the generated data in Table 1 and Table 2, we observed that the obtained solutions for our methods are significantly similar at any well-chosen standard tolerance rate. Furthermore, the choice of the scalar sequence $\{\alpha_n\}$ affects the rate of convergence of the constructed algorithms to a great deal. The results obtained in Tables 1 and 2, together with Figure 1 demonstrates the effectiveness and convergence of our Theorems 3.1 and 3.2. In addition, Theorems 1 and 2 are an extension of the results of Majee and Nahak (2018), in the sense that our results demonstrate strong convergence of the sequences generated by our algorithms in the framework of reflexive Banach spaces as over theirs which demonstrate weak convergence in Hilbert space. It is also an improvement of some results cited in this paper in the sense that our algorithms has the inertial component known to increase/speed up convergence over algorithms without the inertial components (see Polyak (1964), Chidume (2018) and the references it them).

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