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An efficient 4-step block method for solution ff first order initial value problems via shifted chebyshev polynomial

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Abstract

In this paper, we develop a four-step block method for solution_of first order initial value problems of ordinary differential equations. The collocation and interpolation approach is adopted to obtain a continuous scheme for the derived method via Shifted Chebyshev Polynomials, truncated after sufficient terms. The properties of the proposed scheme such as order, zero-stability, consistency and convergence are also investigated. The derived scheme is implemented to obtain numerical solutions of some test problems, the result shows that the new scheme competes favorably with exact solution and some existing methods.

Keywords: Interpolation, Collocation, Block method, Zero-stability, Convergence

1. Introduction

It is a general knowledge that differential equations model quite a lot of physical problems that occur in many fields of Sciences, Mathematics and Engineering; common among such differential equations are ordinary differential equations. Thus, the solutions to these differential equations and consequently the physical problems that they represent, have been a subject of interest to Mathematicians, particularly Numerical Analysts. A wide range of numerical schemes for solving initial value problems of first order ordinary differential equation using different approaches have been developed and are continuously being sort, with extension to higher order ordinary differential equations. Listing just a few of such research would include Lambert (1991), Lambert (1973), Ayinde and Ibijola (2015), Taiwo (2005), Ogunrinde *et al* (2012), Fadugba and Idowu (2019), Fadugba and Falodun (2017), Fadugba and Okunlola (2017), Fadugba *et al* (2020), Aksah *et al* (2019), Sunday *et al* (2013), Suleiman *et al* (2013), Adeyeye and Kayode (2013), Odekunle *et al* (2012), Awoyemi (2014), Fatunla (1991), Yahaya (2007), Muhammed and Yahaya (2010), Adeniyi *et al* (2006), Aboiyar *et al* (2015), Olanegan *et al* (2016), Adeniyi and Alabi (2006), Onumanyi *et al* (1994) among other numerous numerical integrators for the solution of initial value problems of various orders. Many of the above mentioned methods employed interpolation and collocation approach in deriving schemes of varying accuracy for first order initial value problems. Hence, the evolution of ordinary differential equation as well as its properties and solvability has been and is still being extensively researched. The need to employ suitable polynomials which approximates a function accurately and uniformly on some interval [a, b] is unceasing, hence the consideration of minimax approximants. Although

minimax polynomials exist and are unique when a function is continuous, they are not easy to compute in general. Therefore, a more effective approach is to consider near-minimax polynomials such as Chebyshev polynomials. However, Shifted Chebyshev polynomials are more suitable when the range of the independent variable is [0, 1] instead of [−1, 1].
The general definition of the Shifted Chebyshev polynomials is given as

$$T_n^*(x) = T_n(2x-1)\,, \qquad 0 \le x \le 1 \tag{1}$$

In what follows, we shall develop a class of a four step linear multistep method using the Shifted Chebyshev polynomial as basis function and collocation and interpolation approach for the derivation of the method. The rest of this paper is divided into the following sections: section 2 gives the derivation of the method, sections 3 comprises of the analysis of the properties of the derived scheme to include order, error constants, zero-stability and convergence of the method, implementation of the method on some test problems is done in section 4 while section 5 is discussion and conclusion.

2. Method derivation

Consider the general form of a first order initial value problem:

$$y' = f(x, y(x)) \quad y(a) = y_0, \; a \le x \le b, \tag{2}$$

a and b being finite.

We assume that (2) has a solution, $y(x)$ of the form:

$$y(x) = \sum_{i=0}^{s+m-1} T_j^* a_i (x - x_n) \tag{3}$$

(3) is evaluated at x_n and x_{n+4}, which are chosen as interpolation points to give:

$$y(x_n) = a_0 - a_1 + a_2 - a_3 + a_4 - a_5 \tag{4}$$

$$y(x_{n+4}) = a_0 + a_1(8h-1) + a_2(128h^2 - 32h + 1) + a_3(2048h^3 - 768h^2 + 72h - 1) + a_4(32768h^4 - 16384h^3 + 2560h^2 - 128h + 1) + a_5(524288h^5 - 327680h^4 + 71680h^3 - 6400h^2 + 200h - 1) \tag{5}$$

Also, the first derivative of (3) is obtained as:

$$y'(x) = f(x, y(x)) = \sum_{i=1}^{s+m-1} T_j^* a_i (x - x_n) \tag{6}$$

(6) is then evaluated at x_n, x_{n+1}, x_{n+2} and x_{n+3}, considered as collocation points to obtain:

$$f(x_n) = 2a_1 - 8a_2 + 18a_3 - 32a_4 + 50a_5 \tag{7}$$

$$f(x_{n+1}) = 2a_1 + a_2(16h-8) + a_3(96h^2 - 96h + 18) + a_4(512h^3 - 768h^2 + 320h - 32) + a_5(2560h^4 - 5120h^3 + 3360h^2 - 800h + 50) \tag{8}$$

$$f(x_{n+2}) = 2a_1 + a_2(32h - 8) + a_3(384h^2 - 192h + 18) + a_4(4096h^3 - 3072h^2 + 640h - 32) + a_5(40960h^4 - 40960h^3 + 13440h^2 - 1600h + 50) \quad (9)$$

$$f(x_{n+3}) = 2a_1 + a_2(48h - 8) + a_3(864h^2 - 288h + 18) + a_4(13824h^3 - 6912h^2 + 960h - 32) + a_5(207360h^4 - 138240h^3 + 30240h^2 - 2400h + 50) \quad (10)$$

(4) , (5), (7), (8), (9) and (10) form a system of six equations in six unknowns (a_0, a_1, a_2, a_3, a_4 and a_5) which result from the interpolation and collocation of the basis function (3). This system of equations is then solved to obtain the values for the unknown coefficients.

We recall that:

$$\left.\begin{aligned} x_n &= x_0 + nh \\ x_{n+j} &= x_n + jh \\ y(x_n) &= y_n \\ y(x_{n+j}) &= y_{n+j} \\ f(x_n, y(x_n)) &= f_n \\ f(x_{n+j}, y(x_{n+j})) &= f_{n+j} \end{aligned}\right\} \quad (11)$$

Combining the solutions obtained in solving the system of equations in (4), (5), (7), (8), (9) and (10) with (11) and substituting back into (3), the continuous form of the proposed 4-step method is obtained as:

$$y(x) = \sum_{j=0}^{j=4} a_j(x) y_{n+j} - h \sum_{j=0}^{j=4} \beta_j(x) f_{n+j} \quad (12)$$

where a_j and β_j as obtained are:

$$\left.\begin{aligned} a_0(x) &= 1 + \frac{45(x-x_n)^2}{112h^2} - \frac{55(x-x_n)^3}{112h^3} + \frac{45(x-x_n)^4}{224h^4} - \frac{3(x-x_n)^5}{112h^5} \\ a_4(x) &= \frac{45(x-x_n)^2}{112h^2} + \frac{55(x-x_n)^3}{112h^3} - \frac{45(x-x_n)^4}{224h^4} + \frac{3(x-x_n)^5}{112h^5} \\ \beta_0(x) &= (x - x_n) - \frac{11(x-x_n)^2}{12h} + \frac{(x-x_n)^3}{3h^2} - \frac{(x-x_n)^4}{24h^3} \\ \beta_1(x) &= \frac{18(x-x_n)^2}{7h} - \frac{15(x-x_n)^3}{7h^2} + \frac{37(x-x_n)^4}{56h^3} - \frac{(x-x_n)^5}{14h^4} \\ \beta_2(x) &= -\frac{9(x-x_n)^2}{7h} + \frac{37(x-x_n)^3}{28h^2} - \frac{11(x-x_n)^4}{28h^3} + \frac{(x-x_n)^5}{28h^4} \\ \beta_3(x) &= \frac{26(x-x_n)^2}{21h} - \frac{31(x-x_n)^3}{21h^2} + \frac{97(x-x_n)^4}{168h^3} - \frac{(x-x_n)^5}{14h^4} \end{aligned}\right\} \quad (13)$$

We evaluate (12) at $x = x_n,\ x_{n+1},\ x_{n+2}$ and x_{n+3} while the derivative of (12) is evaluated at x_{n+4} to yield the following four discrete schemes:

$$\begin{aligned} 19y_{n+4} + 224y_{n+1} - 243y_n &= h(60f_{n+3} - 72f_{n+2} + 288f_{n+1} + 84f_n \\ 3y_{n+4} + 84y_{n+2} - 87y_n &= h(8f_{n+3} + 24f_{n+2} + 120f_{n+1} + 28f_n \\ 27y_{n+4} + 224y_{n+3} - 251y_n &= h(156f_{n+3} + 216f_{n+2} + 324f_{n+1} + 84f_n \\ 45y_{n+4} - 45y_n &= h(14f_{n+4} + 64f_{n+3} + 24f_{n+2} + 64f_{n+1} + 14f_n \end{aligned} \quad (14)$$

The discrete schemes presented in (14) gives the proposed 4-step block method (4SBM) which directly integrates general first order initial value problems of ordinary differential equation. The derived method is self-starting and is thus potentially suitable and easy to implement as numerical solver for initial value problem of ordinary differential equations.

3.0 Analysis of the Derived 4SBM

3.1 Order of 4SBM

According to Lambert (1973), the linear difference operator associated with (12) is defined by

$$L[y(x); h] = \sum[\alpha_j y(x_n + jh) - h\beta_j f(x_n + jh)] \quad (15)$$

where $y(x)$ is assumed to have continuous derivative of sufficiently high order.

The Taylor series expansion of (15) gives:

$$L[y(x);h] = c_0 y(x) + c_1 h y'(x) + c_2 h^2 y''(x) + \cdots + c_p h^p y^{(p)}(x) + c_{p+1} h^{p+1} y^{(p+1)}(x) + c_{p+2} h^{p+2} y^{(p+2)}(x) + \cdots \quad (16)$$

where c_o, c_1, …, c_p are defined as:

$$\left.\begin{aligned} c_0 &= \alpha_0 + \cdots + \alpha_k \\ c_1 &= \sum_{j=0}^{k} j\,\alpha_j \\ &and \\ c_p &= \frac{1}{p!}\sum_{j=1}^{k} j^p \alpha_j - \frac{1}{(p-1)!}\sum_{j=1}^{k} j^{(p-1)} \beta_j \quad p \ge 2 \end{aligned}\right\} \quad (17)$$

The derived method is said to be of order p and error constant c_{p+1} if

$$c_0 = c_1 = \cdots = c_p = 0, \ c_{p+1} \neq 0. \quad (18)$$

We now determine the $c_p's$ for each of the four discrete schemes in the derived block method in (14) above as follows:

$$\left.\begin{aligned} c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = 0,\ c_6 = \frac{21}{5} \\ c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = 0,\ c_6 = \frac{14}{15} \\ c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = 0,\ c_6 = \frac{21}{5} \\ c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 0,\ c_7 = \frac{-8}{21} \end{aligned}\right\} \quad (19)$$

In light of (19), it can be deduced that the four discrete schemes in the derived 4SBM method are of non-uniform order $(5, 5, 5, 6)^T$ with error constants:

$$c_{p+1} = \left[\frac{21}{5}, \frac{14}{15}, \frac{21}{5}, \frac{-8}{21}\right] \quad (20)$$

3.2 Consistency and Zero Stability of 4SBM

According to Adeniyi *et al* (2006), a linear multistep method is said to be consistent if it is of order greater than or equal to 1.

Since the derived method is of order greater than one, we conclude that the method is consistent.

Also, as stated in Henrici (1962), a linear multistep method is zero-stable for any well behaved initial value problem if

- all roots of the first characteristic polynomial, $\rho(r)$ lies within the unit circle, $|\, r \,| \le 1$.
- any roots of $\rho(r)$ on the unit circle ($|\, r \,| = 1$) are simple.

In addition, $(1) = 0$, $\rho'(1) = \sigma(1)$. where $\sigma(r)$ is the second characteristic polynomial.

The first and second characteristic polynomials are defined respectively as:

$$\left.\begin{aligned} \rho(r) &= \sum_{j=0}^{k} \alpha_j r^j \\ \sigma(r) &= \sum_{j=0}^{k} \beta_j r^j \end{aligned}\right\} \quad (21)$$

Obtaining the first characteristic polynomials of the discrete schemes in the derived 4SBM in (14) gives:

$$\left.\begin{aligned} \rho_1(r) &= 19r^4 + 224r - 243 \\ \rho_2(r) &= 3r^4 + 84r^2 - 87 \\ \rho_3(r) &= 27r^4 + 224r^3 - 251 \\ \rho_4(r) &= 45r^4 - 45 \end{aligned}\right\} \quad (22)$$

Also, the second characteristic polynomials are obtained as:

$$\left.\begin{aligned} \sigma_1(r) &= 60r^3 - 72r^2 + 288r + 84 \\ \sigma_2(r) &= 8r^3 + 24r^2 + 120r + 28 \\ \sigma_3(r) &= 156r^3 + 216r^2 + 324r + 84 \\ \sigma_4(r) &= 14r^4 + 64r^3 + 24r^2 + 64r + 14 \end{aligned}\right\} \quad (23)$$

We note that $\rho_4(r)$ in (22) and $\sigma_4(r)$ in (23) represents the first and second characteristic polynomials of the main scheme respectively which we denote as $\rho(r)$ and $\sigma(r)$.

Furthermore, the roots of the first characteristic polynomial are given as:

$$\rho(r) = [1,\ -1,\ i,\ -i\,] \quad (24)$$

i.e. the roots of the $\rho(r)$ lies within the unit circle and all roots with modulus 1 are simple.

It is equally observed that $\rho(1) = 0$ and $\rho'(1) = \sigma(1)$ which also holds for all the other three schemes in the derived method.

Consequently, the 4SBM method is considered to be zero stable.

3.3 Convergence of 4SBM

Following Lambert (1991), the necessary and sufficient conditions for convergence are consistency and stability. The derived method was shown to be consistent and zero-stable, hence it is convergent.

3.4 Region of Stability

Consider the test problem:

$$y' = -\lambda y, \quad y(0) = 1 \quad (25)$$

where λ is a complex constant.

We apply (25) to each of the discrete schemes in (14) and plot the respective regions of stability for the schemes using ***MATLAB R2016a*** software as given in the figures below:

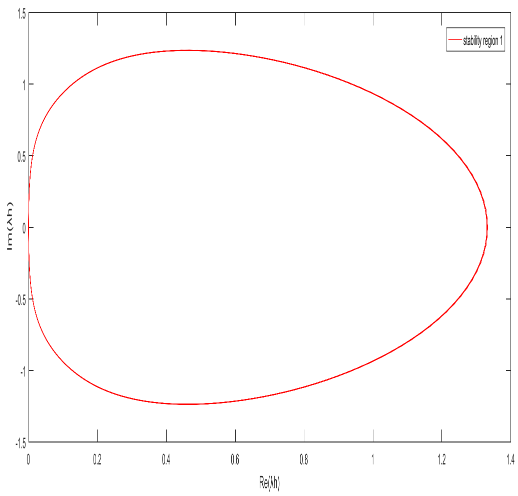

Fig 1: Stability Region for Scheme 1

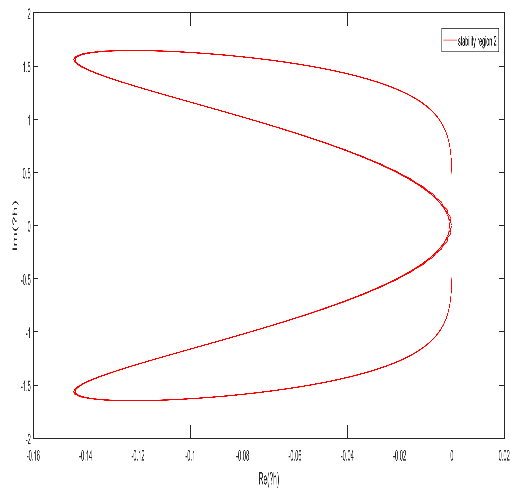

Fig 2: Stability Region for Scheme 2

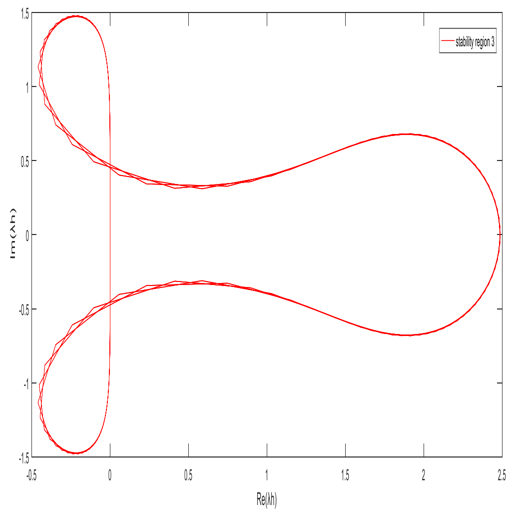

Fig 3: Fig 1: Stability Region for Scheme 3

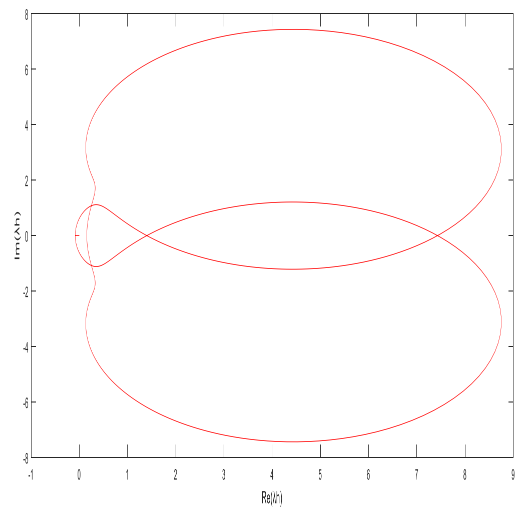

Fig 4: Fig 1: Stability Region for Scheme 4

4. Results

4.1 Numerical results

In this section, we examine the performance of 4SBM in terms of accuracy and stability when compared with exact solution (ES). Also, a comparative study of 4SBM and some existing methods in the form of Fadugba *et al* (2020) and Aksah *et al* (2019) is presented. All numerical computations were carried out with the aid of **MAPLE 17**, 64bits software.

The following notations are used:

SCNM : Second Order Numerical Method… by Fadugba *et al* (2020)

SDIBBDF : Singly diagonally implicit BBDF method by Aksah *et al* (2019)

Problem 1

$$\frac{dy}{dx} - x\sin x = 0, \quad y(0) = 1, \quad x \in [0,1]$$

The exact solution to problem 1 is given by:

$y(x) = 1 + \sin x - x\cos x$

Source: Fadugba *et al* (2020)

Problem 2

$$\frac{dy}{dx} - 1 - (y - x)^2 = 0, \quad y(0) = \frac{1}{2}, \quad x \in [0,1]$$

Exact solution: $y(x) = \frac{(x+1)^2}{x+2}$

Source: Fadugba *et al* (2020)

Problem 3

We consider the Riccati equation coupled with initial condition as given below:

$$\frac{dy}{dx} + 4 - 4y + y^2 = 0, \quad y(0) = 1, \quad x \in \left[0, \frac{2}{5}\right]$$

with exact solution: $y(x) = \frac{2x-1}{x-1}$ having a pole at $x = 1$.

Source: Fadugba *et al* (2020)

Problem 4

Consider the stiff initial value problem:

$$y' = -20y + 20\sin x + \cos x, \quad y(0) = 1, \quad x \in [0,1]$$

Exact Solution: $y(x) = \sin x + e^{-20x}$,

Eigenvalue: $\lambda = -20$

Source: Aksah *et al* (2019)

Each of problems 1 to 4 is solved with the derived 4SBM in their respective intervals with $h = 0.05$ and the results presented alongside exact solutions and absolute errors in 4SBM in the succeeding tables.

Table 1: Results Obtained via 4SBM and Exact Solution for Problem 1

x	$4SBM$	ES	$Error$
0.05	1.000041656098423	1.000041656250930	1.525 E-10
0.10	1.000333000034864	1.000333000119025	8.416 E-11
0.15	1.001122470624527	1.001122470783193	1.587 E-10
0.20	1.002656015266213	1.002656015226813	3.940 E-11
0.25	1.005175853371282	1.005175853826862	4.556 E-10
0.30	1.008919259675645	1.008919259923658	2.480 E-10
0.35	1.014117357497459	1.014117357958868	4.614 E-10
0.40	1.020993944784203	1.020993944707496	7.671 E-11
0.45	1.029764337318100	1.029764338052525	7.344 E-10
0.50	1.040634257260341	1.040634257659017	3.987 E-10
0.55	1.053798741058314	1.053798741797931	7.397 E-10
0.60	1.069441104559158	1.069441104449228	1.099 E-10
0.65	1.087731935705026	1.087731936679154	9.741 E-10
0.70	1.108828155610459	1.108828156138549	5.280 E-10
0.75	1.132872107389567	1.132872108367968	9.784 E-10
0.80	1.159990723559075	1.159990723421791	1.373 E-10
0.85	1.190294729976340	1.190294731138058	1.162 E-09
0.90	1.223877937554632	1.223877938183885	6.293 E-10
0.95	1.260816568633841	1.260816569798685	1.165 E-09
1.00	1.301168679097036	1.301168678939756	1.573 E-10

Table 2: Results Obtained via 4SBM and Exact Solution for Problem 2

x	$4SBM$	ES	$Error$
0.05	0.537804876867552	0.537804878048781	1.181 E-09
0.10	0.576190475537360	0.576190476190476	6.531 E-10
0.15	0.615116277982668	0.615116279069767	1.087 E-09
0.20	0.654545454626120	0.654545454545455	8.067 E-11
0.25	0.694444443897402	0.694444444444444	5.470 E-10
0.30	0.734782608422214	0.734782608695652	2.734 E-10
0.35	0.775531914385746	0.775531914893617	5.079 E-10
0.40	0.816666666773689	0.816666666666667	1.070 E-10
0.45	0.858163265060936	0.858163265306122	2.452 E-10
0.50	0.899999999904091	0.900000000000000	9.591 E-11
0.55	0.942156862515474	0.942156862745098	2.296 E-10
0.60	0.984615384726837	0.984615384615385	1.115 E-10

0.65	1.027358490470479	1.027358490566038	9.556 E-11
0.70	1.070370370359789	1.070370370370370	1.058 E-11
0.75	1.113636363545846	1.113636363636364	9.052 E-11
0.80	1.157142857249961	1.157142857142857	1.071 E-10
0.85	1.200877192962900	1.200877192982456	1.956 E-11
0.90	1.244827586237433	1.244827586206897	3.054 E-11
0.95	1.288983050828196	1.288983050847458	1.926 E-11
1.00	1.333333333432874	1.333333333333333	9.954 E-11

Table 3: Results Obtained via 4SBM and Exact Solution for Problem 3

x	$4SBM$	ES	$Error$
0.05	0.947368868660920	0.947368421052632	4.476 E-07
0.10	0.888889196924966	0.888888888888889	3.080 E-07
0.15	0.823529956981598	0.823529411764706	5.452 E-07
0.20	0.750000100472588	0.750000000000000	1.005 E-07
0.25	0.666669415808271	0.666666666666667	2.749 E-06
0.30	0.571430589912683	0.571428571428571	2.018 E-06
0.35	0.461542012435517	0.461538461538462	3.551 E-06
0.40	0.333334332177455	0.333333333333333	9.988 E-07

Table 4: Results Obtained via 4SBM and Exact Solution for Problem 4

x	$4SBM$	ES	$Error$
0.05	0.415973405593861	0.417858610442121	1.885 E-03
0.10	0.235280102523366	0.235168699883441	1.114 E-04
0.15	0.198429486948619	0.199225200841463	7.957 E-04
0.20	0.218842241432046	0.216984969683795	1.857 E-03
0.25	0.254787128316857	0.254141906253608	6.452 E-04
0.30	0.298252560545128	0.297998958838006	2.536 E-04
0.35	0.343886105695109	0.343809689421006	7.642 E-05
0.40	0.389825288561599	0.389753804936553	7.148 E-05
0.45	0.435114474159927	0.435088943915317	2.553 E-05
0.50	0.479480658115133	0.479470938533965	9.720 E-06
0.55	0.942156862515474	0.522703930631449	3.235 E-06
0.60	0.564650682571834	0.564648617607389	2.064 E-06
0.65	0.605189410336500	0.605188666065447	7.443 E-07
0.70	0.644218799129693	0.644218518766410	2.804 E-07
0.75	0.681639162254277	0.681639065925655	9.633 E-08
0.80	0.717356256350970	0.717356203434697	5.292 E-08
0.85	0.751280465803523	0.751280446539670	1.926 E-08
0.90	0.783326932020669	0.783326924857463	7.163 E-09
0.95	0.813415512958577	0.813415510392170	2.566 E-09
1.00	0.841470987962095	0.841470986869050	1.093 E-09

Also, the maximum errors via applying 4SBM, SCNM and SDIBBDF to solve Problems 1 to 4 in their respective integration interval with varied values of step length given as: $h = 0.5,\ 0.25,$ $0.125,\ 0.0625$ and 0.03125 are obtained. The results are given in the next three tables:

Table 5: Maximum Error in 4SBM, SCNM and SDIBBDF for Problem 1

h	$Maximum$	$Error$	
	$4SBM$	$SCNM$	SDIBBDF
0.05	1.1648432348 E-09	9.1679143038 E-04	1.2872612859 E-02
0.025	1.7545844683 E-11	2.3039010417 E-04	6.7243621653 E-03
0.0125	2.9739429002 E-13	6.5412117264 E-07	3.4341600817 E-03
0.00625	4.7368321743 E-15	1.6251493433 E-07	1.7350726080 E-03
0.003125	7.3317461306 E-17	3.6162173629 E-08	8.7203430942 E-04

Table 6: Maximum Error in 4SBM, SCNM and SDIBBDF for Problem 2

h	$Maximum$	$Error$	
	$4SBM$	$SCNM$	SDIBBDF
0.05	1.08709943881 E-09	1.08543181681 E-05	1.6071672413 E-03
0.025	2.17329064171 E-11	2.64907317363 E-06	6.1855030453 E-04
0.0125	3.69327425795 E-13	6.54121172638 E-07	3.8188162246 E-04
0.00625	6.02183095715 E-15	1.62520328014 E-07	1.8932254244 E-04
0.003125	9.61310579295 E-17	4.05043625252 E-08	9.4258783854 E-05

Table 7: Maximum Error in 4SBM, SCNM and SDIBBDF for Problem 3

h	$Maximum$	$Error$	
	$4SBM$	$SCNM$	SDIBBDF
0.05	3.5508970546 E-06	5.0224441532 E-03	3.6129445482 E-02
0.025	8.3887224623 E-08	1.3518201515 E-03	2.0630450070 E-02
0.0125	1.6471472949 E-09	3.5047481168 E-04	1.1036160711 E-02
0.00625	2.1872547934 E-11	8.9209176545 E-05	5.7105713993 E-03
0.003125	3.1582112998 E-13	2.2502483560 E-05	2.9051182479 E-03

Table 8: Maximum Error in 4SBM, SCNM and SDIBBDF for Problem 4

h	$Maximum$	$Error$	
	$4SBM$	$SCNM$	SDIBBDF
0.05	3.5508970546 E-06	6.5870995837 E-01	6.6243451748 E-04
0.025	8.3887224623 E-08	6.5857260320 E-01	3.3472344598 E-04
0.0125	1.6471472949 E-09	6.5853971240 E-01	1.6829289554 E-04
0.00625	2.1872547934 E-11	6.5853166363 E-01	8.4384411578 E-05
0.003125	3.1582112998 E-13	6.5787969735 E-01	4.2252348871 E-05

The maximum errors presented in tables 5 to 8 are visualized in figures 5 to 8 respectively to further illustrate the superiority of 4SBM over the existing methods under consideration.

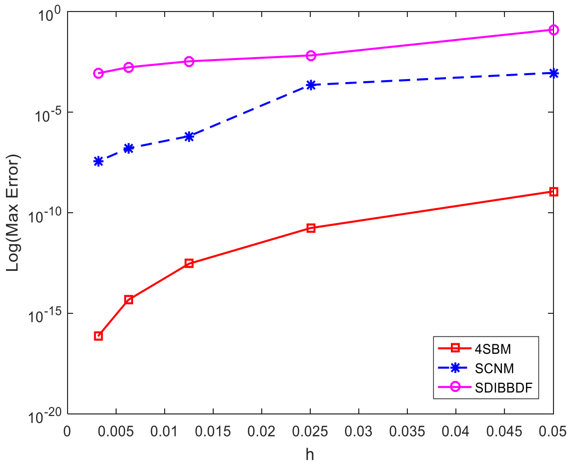

Fig 5: Comparison of Errors for Problem 1

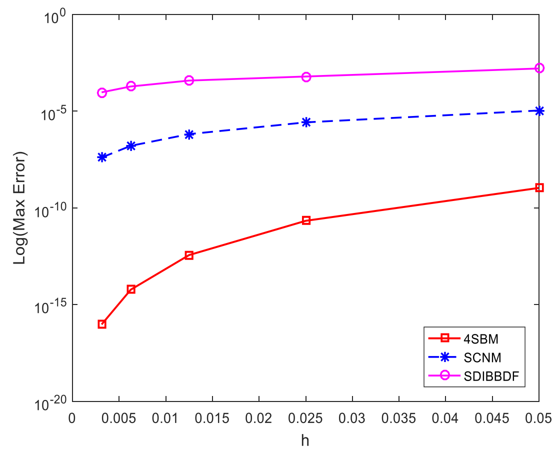

Fig 6: Comparison of Errors for Problem 2

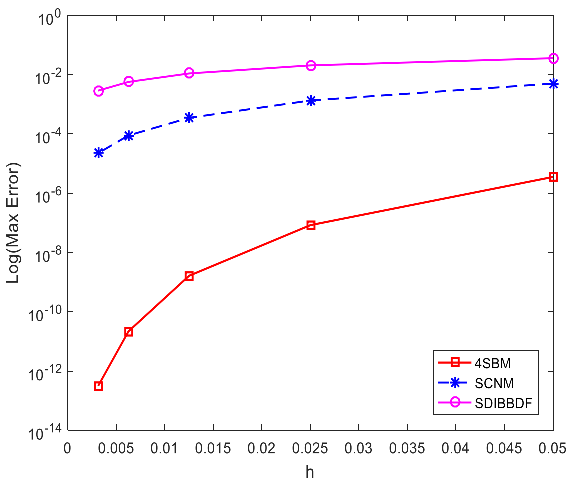

Fig 7: Comparison of Errors for Problem 3

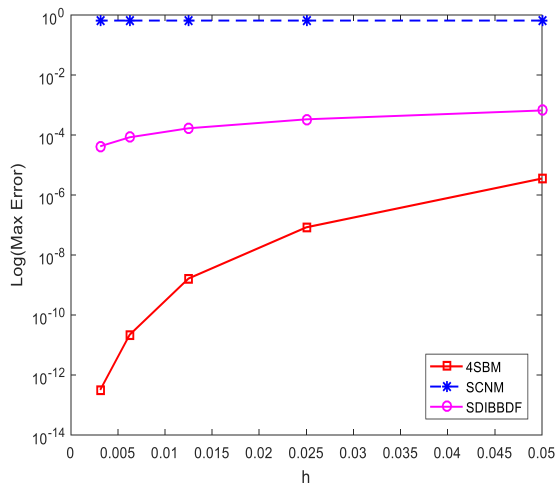

Fig 8: Comparison of Errors for Problem 4

5. Discussion

In this paper, we developed a 4 step block linear multistep (4SBM) method for solving IVPs in ODEs. The scheme was derived via Chebyshev Polynomial as the interpolating function. The properties of the 4SBM were extensively discussed. It is clear from the analysis of the properties of the derived 4SBM that the method is consistent, stable, and convergent with the discrete schemes in the method having non-uniform order of accuracy, the minimum of which was five. Four numerical examples were solved successfully as test problems in order to evaluate the performance of 4SBM in terms of accuracy and stability. Tables 1-4 shows the results obtained when solving the four test problems in their respective integration intervals presented alongside exact solutions. It can be seen that the 4SBM agreed with the exact solutions to a quite appreciable extent. Furthermore, a comparative study of 4SBM, SOCNM and SDIBBDF was also presented. By varying the step length, the maximum errors obtainable within the associated integration interval for each problem was computed.

6. Conclusion

The results generated from the comparative study of the 4SBM, SOCNM and SDIBBDF are given in tables 5 to 8 and displayed graphically in figures 5 through 8. It can be observed that the derived 4SBM competes favourably with the other methods under review. While it is quite obvious from the numerical results in tables 5 to 8 that 4SBM is more accurate than its equals, it is equally visible from the maximum error curves in Figures 5, 6, 7 and 8 that there is an impressive deterioration in the errors generated via 4SBM as the step size decreases, a confirmation of the suitability of the derived method to the model problems. Consequently, we conclude that the derived 4SBM is accurate, stable and hence a good self-starting numerical solver for IVPs of different characteristics in ODEs.

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